Reducing Acyclic Network Coding Problems to Single-Transmitter-Single-Demand Form*

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Abstract

Médard et al [1] suggest that any acyclic network coding problem can be reduced to an equivalent Single-Transmitter-Single-Demand form, thus enabling consideration of a structured subclass of networks without loss of generality.

We show that the prior construction for deriving STSD networks only holds when information sources available to different transmitter nodes are independent. We complete the proof of equivalence using the prior construction under this additional independence condition. We propose a new construction for deriving STSD equivalent forms that works for networks with arbitrarily correlated sources. Our results enable the application of STSD equivalence to a larger class of network coding problems.

1 Introduction

Network coding concerns the theory and design of coding strategies carried out by nodes in a network to achieve desired communication. For an introduction to network coding and network information flow, the reader is referred to the book by Yeung [2], or papers [3] [4].

Médard et al. [1] argue that for any acyclic network coding problem there exists an equivalent problem in Single-Transmitter-Single-Demand (STSD) form, such that the original problem is achievable if and only if the STSD problem is achievable. This enables us to study a subset of structured networks without loss of generality. In [1], the STSD equivalent problem is systematically constructed and its equivalence to the original problem argued.

In this paper, we show that the construction in [1] is limited to the case where information sources available at different nodes are independent. We make clear this limitation by showing an example where the construction breaks down due to source dependency and by filling in the missing proof of equivalence between the derived STSD problem and the original, making explicit use of the independency condition. We then propose a new construction for deriving STSD form equivalent problem under general conditions, allowing arbitrary correlation among information sources. We prove that the STSD form problem derived using the new construction is equivalent to the original one, hence effectively expand the applicability of equivalent STSD forms to a larger class of problems.

The paper is arranged as follows. We describe our system model and define STSD form in Section 2. In Section 3 we review prior work, formulating the STSD equivalence result and the construction used in [1]. Limitations of the prior construction are discussed in Section 4, and the proposed construction is presented in Section 5. Section 6 concludes the paper.

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2 System Model

We adopt a model much similar to that used in [1]. A *network coding problem*, or *network*, $\mathcal{N}$, is denoted by the 5-tuple $\mathcal{N} = (\mathcal{G}, \mathcal{R}, \mathcal{K}, \mathcal{S}, \mathcal{D})$, where

- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a directed graph or multigraph, where $\mathcal{V}$ is the set of *vertices* (a.k.a. *nodes*) and $\mathcal{E}$ is the set of *edges* (a.k.a. *links*);
- $\mathcal{R} = [r(e)]_{e:e\in\mathcal{E}}$ with $r : \mathcal{E} \mapsto \mathbb{R}^+$ defines *edge capacities*;
- $\mathcal{K}$ is the index set of *information sources*, which are denoted by $B_k, k \in \mathcal{K}$;
- $\mathcal{S} = [s(k,v)]_{k\in\mathcal{K},v\in\mathcal{V}}$ and $\mathcal{D} = [d(k,v)]_{k\in\mathcal{K},v\in\mathcal{V}}$ are $|\mathcal{K}| \times |\mathcal{V}|$ matrices that describe the communication scenario, where

$$s(k,v) = \begin{cases} 1 & \text{if } B_k \text{ is available at node } v \text{ (for transmission)} \\ 0 & \text{otherwise} \end{cases},$$

$$d(k,v) = \begin{cases} 1 & \text{if } B_k \text{ is demanded by node } v \text{ (to be received)} \\ 0 & \text{otherwise} \end{cases}.$$

Unless otherwise stated, for the rest of this paper we make the following assumptions:

1. $\mathcal{G}$ is acyclic (in which case we shall also say that $\mathcal{N}$ is *acyclic*).
2. All information sources are independent[1] binary Bernoulli$(1/2)$ random sources with information rate of 1 bit/time unit.
3. Over any edge e, transmission of information at rates no greater than $r(e)$ is error-free, while transmission at rates higher than $r(e)$ is prohibited.
4. Without loss of generality, let $\mathcal{K} = \{1, 2, \ldots, K\}$, where $K \stackrel{\text{def}}{=} |\mathcal{K}|$.
5. $\forall k \in \mathcal{K}, \sum_{v\in\mathcal{V}} d(k,v) > 0$, and $\forall k \in \mathcal{K}, \forall v \in \mathcal{V}, s(k,v) + d(k,v) \le 1$.

Since with the above assumptions $\mathcal{K}$ is implicitly known, in most cases we will skip writing it out and refer to a network as $\mathcal{N} = (\mathcal{G}, \mathcal{R}, \mathcal{S}, \mathcal{D})$.

Let $B_k^n \stackrel{\text{def}}{=} (B_k(1), B_k(2), \ldots, B_k(n))$ denote the first n samples of B_k and b_k^n a realization of B_k^n. For any $e \in \mathcal{E}$, let e be incident from $t(e)$ to $h(e)$. We denote e as the ordered pair $(t(e), h(e))$. Let $W_e(t)$ be the random process transmitted on edge e. Define $W_e^n \stackrel{\text{def}}{=} (W_e(1), W_e(2), \ldots, W_e(n))$. Any realization of W_e^n, w_e^n, satisfies the capacity constraint of edge e such that $w_e^n \in \mathcal{W}_e^n \stackrel{\text{def}}{=} \{0,1\}^{nr(e)}$. For any $v \in \mathcal{V}$, let $I(v) = \{e : e \in \mathcal{E}, h(e) = v\}$ and $O(v) = \{e : e \in \mathcal{E}, t(e) = v\}$ denote the sets of incoming and outgoing edges of v, respectively. Since $I(v)$ and $O(v)$ are graph specific, we will write $I(v, \mathcal{G})$ and $O(v, \mathcal{G})$ where confusion may arise. Define the sets of sources and demands of node v as $\mathcal{K}_v^s \stackrel{\text{def}}{=} \{k : k \in \mathcal{K}, s(k,v) = 1\}$ and $\mathcal{K}_v^d \stackrel{\text{def}}{=} \{k : k \in \mathcal{K}, d(k,v) = 1\}$, respectively. Also, denote by $\mathcal{V}_s$ the set of nodes with sources (*transmitter* nodes) and $\mathcal{V}_d$ those with demands (*receiver* nodes). That is, $\mathcal{V}_s \stackrel{\text{def}}{=} \{v : v \in \mathcal{V}, \sum_{k=1}^K s(k,v) > 0\}$, $\mathcal{V}_d \stackrel{\text{def}}{=} \{v : v \in \mathcal{V}, \sum_{k=1}^K d(k,v) > 0\}$. Let $X_v(t)$ be the random process of all information available to node v, which consists of information received from other nodes as well as those native to v. Define $X_v^n \stackrel{\text{def}}{=} (X_v(1), X_v(2), \ldots, X_v(n))$. Then a realization of X_v^n can be expressed as $x_v^n = ((b_k^n)_{k\in\mathcal{K}_v^s}, (w_e^n)_{e\in I(v)})$. Clearly, $x_v^n \in \mathcal{X}_v^n \stackrel{\text{def}}{=} \{0,1\}^{n|\mathcal{K}_v^s|} \times (\mathcal{W}_e^n)_{e\in I(v)}$.

[1] As will be noted later, our new construction does not require B_k's to be independent.

Definition 2.1 (Block Code). A length-n *block code* for an acyclic network coding problem $\mathcal{N} = (\mathcal{G}, \mathcal{R}, \mathcal{S}, \mathcal{D})$ is the 2-tuple $C^n = (f^n, g^n)$, where $f^n = \{f_e^n\}_{e \in \mathcal{E}}$, $f_e^n : \mathcal{X}_{t(e)}^n \mapsto \mathcal{W}_e^n$ describes encoding on each edge $e \in \mathcal{E}$, and $g^n = \{g_{k,v}^n\}_{1 \le k \le K, v \in \mathcal{V}}$, $g_{k,v}^n : \mathcal{X}_v^n \mapsto \{0,1\}^n$ describes decoding at each node v for each of the K information sources. The code is applied to $\mathcal{N}$ such that $\forall e \in \mathcal{E}$,

$$w_e^n = f_e^n(x_{t(e)}^n) = f_e^n\left(\left((b_k^n)_{k \in \mathcal{K}_{t(e)}^s}, (w_\epsilon^n)_{\epsilon \in I(t(e))}\right)\right), \tag{1}$$

Please note that the recursive expression in (1) is not ambiguous since $\mathcal{G}$ is acyclic.

Definition 2.2 (Achievability). An acyclic network $\mathcal{N}$ is said to be *achievable* if there exists $n \ge 1$ and a length-n block code $C^n = (f^n, g^n)$, such that $\forall v \in \mathcal{V}, \forall k, 1 \le k \le K$,

$$g_{k,v}^n(x_v^n) = d(k,v) b_k^n, \tag{2}$$

in which case we shall call $C^n = (f^n, g^n)$ a *solution* to $\mathcal{N}$.

Note that the above definition is non-trivial only for (k,v) such that $d(k,v) = 1$. For $d(k,v) = 0$ one can simply define $g_{k,v}^n(x_v^n) = 0^n$ to satisfy (2).

Definition 2.3 (STSD). A network coding problem $\mathcal{N} = (\mathcal{G}, \mathcal{R}, \mathcal{S}, \mathcal{D})$ is said to be of *Single-Transmitter-Single-Demand* form, or *STSD*, if it satisfies that $|\mathcal{V}_s| = 1$, and that $\forall v \in \mathcal{V}_d, |\mathcal{K}_v^d| = 1, |I(v)| = 1, |O(v)| = 0$ and $\sum_{e \in I(v)} r(e) = 1$.

In words, a STSD network is one such that there is only one transmitter node that has all the information sources available, and each receiver node requests one source process and has incoming degree 1 and incoming capacity of 1bit/time unit.

3 Prior Work

We briefly review related results in [1] to facilitate later discussion.

The motivation for considering STSD networks lies in the following theorem.

Theorem 3.1. *For any acyclic network $\mathcal{N}$, there exists an STSD network $\tilde{\mathcal{N}}$ such that $\mathcal{N}$ is achievable if and only if $\tilde{\mathcal{N}}$ is achievable.*

The construction used in [1] for deriving STSD equivalent networks is summarized below.

Construction 3.2 (Médard, et al). Given an acyclic network $\mathcal{N} = (\mathcal{G}, \mathcal{R}, \mathcal{S}, \mathcal{D})$, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, construct STSD network $\tilde{\mathcal{N}} = (\tilde{\mathcal{G}}, \tilde{\mathcal{R}}, \tilde{\mathcal{S}}, \tilde{\mathcal{D}})$ as follows:

1. Add a new node v_0;
2. For each $v \in \mathcal{V}_s$, add a directed edge from v_0 to v and denote it as $e_{v_0,v}$;
3. For each $v \in \mathcal{V}_s \cup \mathcal{V}_d$ and each $k \in \mathcal{K}_v^s \cup \mathcal{K}_v^d$, create a new node $\tilde{v}_{k,v}$, and add a directed edge from v to $\tilde{v}_{k,v}$, which is denoted as $\tilde{e}_{k,v}$;
4. Let $\mathcal{V}' = \{v_0\}$, $\mathcal{E}' = \{e_{v_0,v} : v \in \mathcal{V}_s\}$, $\mathcal{V}'' = \{\tilde{v}_{k,v} : v \in \mathcal{V}_s \cup \mathcal{V}_d, k \in \mathcal{K}_v^s \cup \mathcal{K}_v^d\}$, and $\mathcal{E}'' = \{\tilde{e}_{k,v} : v \in \mathcal{V}_s \cup \mathcal{V}_d, k \in \mathcal{K}_v^s \cup \mathcal{K}_v^d\}$. Set $\tilde{\mathcal{V}} = \mathcal{V} \cup \mathcal{V}' \cup \mathcal{V}''$, $\tilde{\mathcal{E}} = \mathcal{E} \cup \mathcal{E}' \cup \mathcal{E}''$,

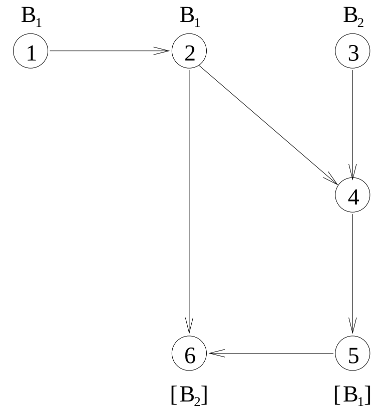

Figure 1: Network not achievable due to contention on edge (4,5)

$\tilde{\mathcal{G}} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$. Set $\tilde{\mathcal{R}} = [\tilde{r}(e)]_{e\in\tilde{\mathcal{E}}}$, $\tilde{\mathcal{S}} = [\tilde{s}(k,v)]_{1\le k\le K, v\in\tilde{\mathcal{V}}}$, and $\tilde{\mathcal{D}} = [\tilde{d}(k,v)]_{1\le k\le K, v\in\tilde{\mathcal{V}}}$, where

$$\begin{aligned} \tilde{r}(e) &= \begin{cases} r(e) & \text{if } e \in \mathcal{E} \\ \sum_{k=1}^{K} s(k, h(e)) & \text{if } e \in \mathcal{E}' \\ 1 & \text{if } e \in \mathcal{E}'' \end{cases}, \\ \tilde{s}(k,v) &= \begin{cases} 1 & \text{if } v = v_0 \\ 0 & \text{otherwise} \end{cases}, \\ \tilde{d}(k,v) &= \begin{cases} 1 & \text{if } v \in \mathcal{V}'' \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Proposition 3.3. *Given an acyclic network $\mathcal{N}$, let $\tilde{\mathcal{N}}$ be the STSD network constructed from $\mathcal{N}$ using Construction 3.2. Then $\mathcal{N}$ is achievable if and only if $\tilde{\mathcal{N}}$ is achievable.*

For a complete proof of Proposition 3.3 and hence of Theorem 3.1, please refer to [1]. Essentially, it is argued that a solution to $\mathcal{N}$ implies a solution to $\tilde{\mathcal{N}}$ and vice versa.

We want to point out that in constructing the solution to $\mathcal{N}$ from a solution to $\tilde{\mathcal{N}}$, the proof in [1] implicitly assumes that in $\tilde{\mathcal{N}}$, the information received by transmitter nodes of $\mathcal{N}$ only depends on the respective sources that are available to them in $\mathcal{N}$. This assumption, as we will see, is generally not true, even for the case in which we shall show that Construction 3.2 holds.

4 Limitations of the Prior Construction

4.1 A Counter-Example

We give an example to demonstrate the idea of Construction 3.2 and to show that it may not yield an equivalent network in certain cases.

Consider the network depicted in Figure 1. All links are of unit capacity. The availability of information sources are marked beside the nodes, and so are the demands, which are denoted in brackets. It is intuitively clear that this network does not have a solution due to contention on edge (4,5). This is easily shown with an entropy argument. However, if we modify the network using Construction 3.2, we obtain the STSD network shown in Figure 2. We see that while the original problem is not achievable, the constructed STSD network is indeed solved

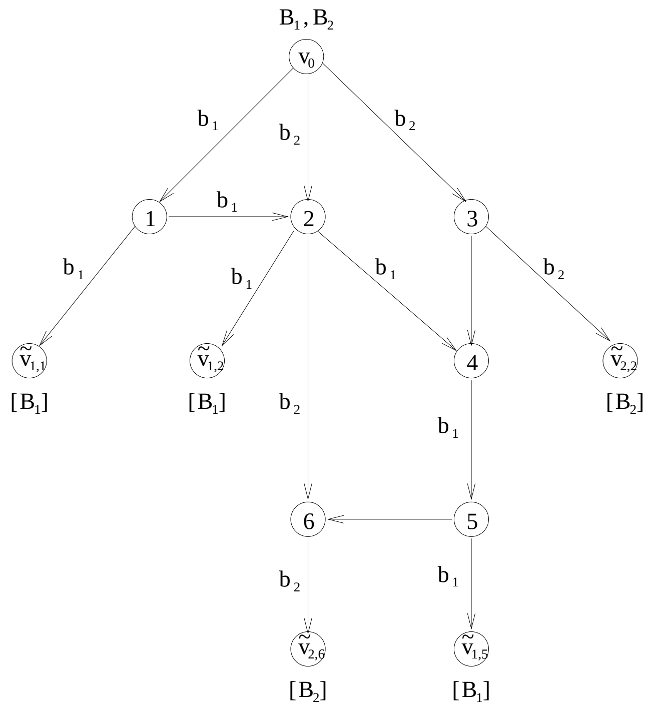

Figure 2: STSD network constructed using Construction 3.2, now with a solution

by a simple length-1 code, also shown in Figure 2, with messages to be transmitted indicated on each edge.

If we apply the proof in [1] to this example, we would assume that if the STSD network is achievable there must exist a solution in which what node 2 receives from v_0 is solely a function of b_1, so that the solution can be "emulated" in the original network. Conceivably, such a solution does not exist here. On the other hand, we have an "unexpected" solution that cannot be emulated.

If we think about how the sample solution we have shown is made possible, the key observation is that when information available to nodes in $\mathcal{V}_s$ are not independent, the total capacity of edges in $\mathcal{V}'$ is over-provisioned, and this leaves open the possibility that a node like node 2 in this example can obtain knowledge of b_1 through other inputs and receive extra information from v_0, which can be impossible to emulate in the original network. This observation motivates our new STSD construction that will be described in Section 5.

4.2 Limited Applicability

We identify the condition under which Construction 3.2 holds.

Theorem 4.1. *For an acyclic network $\mathcal{N}$, let $\tilde{\mathcal{N}}$ be the STSD network constructed from $\mathcal{N}$ using Construction 3.2. If the following is satisfied,*

$$\mathcal{K}_u^s \cap \mathcal{K}_v^s = \emptyset, \quad \forall u, v \in \mathcal{V}_s, u \neq v, \tag{3}$$

then $\mathcal{N}$ is achievable if and only if $\tilde{\mathcal{N}}$ is achievable.

Condition (3) requires that no information source be available at more than one node. Essentially, it requires that information available to different transmitter nodes be independent.

In order to prove Theorem 4.1, we first introduce some terminology and notation. Given a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, define the *parent set* $P(v)$ for $v \in \mathcal{V}$ as $P(v) = \{u : u \in \mathcal{V}, (u, v) \in$

$\mathcal{E}\}$. If $\mathcal{G}$ is acyclic, we can impose a partial ordering on $\mathcal{V}$ such that for any $u, v \in \mathcal{V}$, $u \prec v$ if and only if there is a directed path from u to v. This partial ordering enables us to label nodes in $\mathcal{V}_s$ as $\mathcal{V}_s = \{v_1, v_2, \ldots, v_{|\mathcal{V}_s|}\}$ with the property that if $v_i \prec v_j$ then $i < j$. In the discussion that follows, we assume that such a labelling is already in place.

Now, suppose $\tilde{\mathcal{N}}$ is achievable and a solution $\tilde{C}^n = (\tilde{f}^n, \tilde{g}^n)$ has been applied. For $v \in \mathcal{V}_s$, consider its received message from edge $e_{v_0,v}$, $W^n_{e_{v_0,v}} = \tilde{f}^n_{e_{v_0,v}}(B^n_1, B^n_2, \ldots, B^n_K)$. We now show that $\forall v_i \in \mathcal{V}_s$, $W^n_{e_{v_0,v_i}}$ is a function of $(B^n_k)_{k\in\mathcal{K}^s_{v_i}}$ and $(W^n_e)_{e\in I(v_i,\mathcal{G})}$ by the following two lemmas.[2]

Lemma 4.2. *Let $\tilde{\mathcal{N}}$ be constructed from $\mathcal{N}$ using Construction 3.2. If $\mathcal{N}$ satisfies condition (3) and $\tilde{\mathcal{N}}$ is achievable, then for any $\tilde{C}^n$ that solves $\tilde{\mathcal{N}}$, the random variables $W^n_{e_{v_0,v}}$, $v \in \mathcal{V}_s$ are independent with entropies $H(W^n_{e_{v_0,v}}) = n|\mathcal{K}^s_v|$.*

Proof. Since in $\tilde{\mathcal{N}}$ information sources are only available at v_0 and with $\tilde{C}^n$ all B^n_k's are reconstructed, $(B^n_k)^K_{k=1}$ must effectively be a function of $(W^n_{e_{v_0,v}})_{v\in\mathcal{V}_s}$. Therefore,

$$nK = n\sum_{v\in\mathcal{V}_s}|\mathcal{K}^s_v| \geq \sum_{v\in\mathcal{V}_s} H(W^n_{e_{v_0,v}}) \geq H((W^n_{e_{v_0,v}})_{v\in\mathcal{V}_s}) \geq H((B^n_k)^K_{k=1}) = nK \tag{4}$$

So equality holds throughout, such that $\sum_{v\in\mathcal{V}_s} H(W^n_{e_{v_0,v}}) = H((W^n_{e_{v_0,v}})_{v\in\mathcal{V}_s}) = nK$, and $H(W^n_{e_{v_0,v}}) = n|\mathcal{K}^s_v|$. Hence we conclude that $W^n_{e_{v_0,v}}, v \in \mathcal{V}_s$ are independent, and $H(W^n_{e_{v_0,v}}) = n|\mathcal{K}^s_v|$. □

Please note that the first equality in (4) is due to condition (3).

Lemma 4.3. *Let $\tilde{\mathcal{N}}$ be constructed from $\mathcal{N}$ using Construction 3.2. If $\mathcal{N}$ satisfies condition (3) and $\tilde{\mathcal{N}}$ is achievable, then for any $\tilde{C}^n$ that solves $\tilde{\mathcal{N}}$, $W^n_{e_{v_0,v_i}}$ is a function of $(B^n_k)_{k\in\cup^i_{j=1}\mathcal{K}^s_{v_j}}$ for all $v_i \in \mathcal{V}_s$.*

Proof. We will prove this by induction.

First we show that $W^n_{e_{v_0,v_1}}$ is a function of $(B^n_k)_{k\in\mathcal{K}^s_{v_1}}$. Consider nodes in the parent set of v_1 in the original network $\mathcal{N}$. There cannot be any directed path from nodes in $\mathcal{V}_s$ to those in $P(v_1)$, because $v_1 \not\prec v, \forall v \in P(v_1)$, and by definition $v_i \not\prec v_1$, $i = 2, \ldots, |\mathcal{V}_s|$. Since in $\tilde{\mathcal{N}}$ the only incoming edges added with respect to $\mathcal{V}$ are $\{e_{v_0,v} : v \in \mathcal{V}_s\}$, we instantly conclude that in $\tilde{\mathcal{N}}$ there is no directed path from v_0 to any node in $P(v_1)$. Therefore, nodes in $P(v_1)$ are not sending any messages to v_1, which implies that $(B^n_k)_{k\in\mathcal{K}^s_{v_1}}$ must be a function of $W^n_{e_{v_0,v_1}}$ since

$$B^n_k = \tilde{g}^n_{k,\tilde{v}_{k,v_1}}(\tilde{f}^n_{\tilde{e}_{k,v_1}}(X^n_{v_1})) = \tilde{g}^n_{k,\tilde{v}_{k,v_1}}(\tilde{f}^n_{\tilde{e}_{k,v_1}}(W^n_{e_{v_0,v_1}}, (W^n_e)_{e\in I(v_1,\mathcal{G})})).$$

Thus, we have

$$\begin{aligned} H(W^n_{e_{v_0,v_1}}|(B^n_k)_{k\in\mathcal{K}^s_{v_1}}) &= H(W^n_{e_{v_0,v_1}}, (B^n_k)_{k\in\mathcal{K}^s_{v_1}}) - H((B^n_k)_{k\in\mathcal{K}^s_{v_1}}) \\ &= H(W^n_{e_{v_0,v_1}}) - H((B^n_k)_{k\in\mathcal{K}^s_{v_1}}) \\ &= n|\mathcal{K}^s_{v_1}| - n|\mathcal{K}^s_{v_1}| \\ &= 0 \end{aligned}$$

Now for the general case of $v_i \in \mathcal{V}_s$, it is straightforward to extend the observation we made for v_1 to conclude that what v_i can receive from its incoming edges in $\mathcal{E}$ is always a function

[2] As we commented before, even when condition (3) is satisfied, it is not generally true that $W^n_{e_{v_0,v_i}}$ is a function of $(B^n_k)_{k\in\mathcal{K}^s_{v_i}}$ only.

of $(W^n_{e_{v_0,v_j}})_{j=1}^{i-1}$ as a consequence of the partial ordering. That is, $(W^n_e)_{e\in I(v_i,\mathcal{G})}$ is a function of $(W^n_{e_{v_0,v_j}})_{j=1}^{i-1}$, which in turn is a function of $(B^n_k)_{k\in\cup_{j=1}^{i-1}\mathcal{K}^s_{v_j}}$ by the induction assumption. This implies that $(W^n_e)_{e\in I(v_i,\mathcal{G})}$ and $W^n_{e_{v_0,v_i}}$ are independent because of Lemma 4.2. It also implies that $(W^n_e)_{e\in I(v_i,\mathcal{G})}$ and $(B^n_k)_{k\in\mathcal{K}^s_{v_i}}$ are independent because of condition (3). Finally, noting that $(B^n_k)_{k\in\mathcal{K}^s_{v_i}}$ is a function of $W^n_{e_{v_0,v_i}}$ and $(W^n_e)_{e\in I(v_i,\mathcal{G})}$ since $\tilde{C}^n$ is a solution to $\tilde{\mathcal{N}}$, we have

$$\begin{aligned}
& H(W^n_{e_{v_0,v_i}}|(B^n_k)_{k\in\mathcal{K}^s_{v_i}},(W^n_e)_{e\in I(v_i,\mathcal{G})}) \\
= \;& H(W^n_{e_{v_0,v_i}},(B^n_k)_{k\in\mathcal{K}^s_{v_i}},(W^n_e)_{e\in I(v_i,\mathcal{G})}) - H((B^n_k)_{k\in\mathcal{K}^s_{v_i}},(W^n_e)_{e\in I(v_i,\mathcal{G})}) \\
= \;& H(W^n_{e_{v_0,v_i}},(W^n_e)_{e\in I(v_i,\mathcal{G})}) - H((B^n_k)_{k\in\mathcal{K}^s_{v_i}},(W^n_e)_{e\in I(v_i,\mathcal{G})}) \\
= \;& H(W^n_{e_{v_0,v_i}}) + H((W^n_e)_{e\in I(v_i,\mathcal{G})}) - H((B^n_k)_{k\in\mathcal{K}^s_{v_i}}) - H((W^n_e)_{e\in I(v_i,\mathcal{G})}) \\
= \;& n|\mathcal{K}^s_{v_i}| - n|\mathcal{K}^s_{v_i}| \\
= \;& 0.
\end{aligned}$$

Therefore, $W^n_{e_{v_0,v_i}}$ is a function of $(B^n_k)_{k\in\mathcal{K}^s_{v_i}}$ and $(W^n_e)_{e\in I(v_i,\mathcal{G})}$, which in turn implies that $W^n_{e_{v_0,v_i}}$ is a function of $(B^n_k)_{k\in\cup_{j=1}^{i}\mathcal{K}^s_{v_j}}$. □

Corollary 4.4. *Let $\tilde{\mathcal{N}}$ be constructed from $\mathcal{N}$ using Construction 3.2. If $\mathcal{N}$ satisfies condition (3) and $\tilde{\mathcal{N}}$ is achievable, then for any $\tilde{C}^n$ that solves $\tilde{\mathcal{N}}$, $W^n_{e_{v_0,v_i}}$ is a function of $(B^n_k)_{k\in\mathcal{K}^s_{v_i}}$ and $(W^n_e)_{e\in I(v_i,\mathcal{G})}$ for all $v_i\in\mathcal{V}_s$.*

Proof. This is a direct consequence of the proof of Lemma 4.3. □

With the above developments we are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Given a solution C^n to $\mathcal{N}$, it is straightforward to construct a solution to $\tilde{\mathcal{N}}$ by providing the transmitter nodes of $\mathcal{N}$ with their respective sets of information sources through edges in $\mathcal{E}'$, reusing the edge encoders of C^n for all edges in $\mathcal{E}$ and forwarding (decoded) information bits to nodes in $\mathcal{V}''$. (For a more complete statement please refer to [1].)

We now show that if $\tilde{\mathcal{N}}$ is achievable then so is $\mathcal{N}$. Let $\tilde{C}^n=(\tilde{f}^n,\tilde{g}^n)$ be a solution to $\tilde{\mathcal{N}}$. We want to find a solution $C^n=(f^n,g^n)$ for $\mathcal{N}$. We have shown that when $\tilde{C}^n$ is applied to $\tilde{\mathcal{N}}$, $W^n_{e_{v_0,v_i}}$ is a function of $(B^n_k)_{k\in\mathcal{K}^s_{v_i}}$ and $(W^n_e)_{e\in I(v_i,\mathcal{G})}$ for all $v_i\in\mathcal{V}_s$. Denote this functional relation as

$$W^n_{e_{v_0,v_i}} = \tilde{h}^n_{v_i}((B^n_k)_{k\in\mathcal{K}^s_{v_i}},(W^n_e)_{e\in I(v_i,\mathcal{G})}).$$

For $\mathcal{N}$, define

$$f^n_e(x^n_{t(e)}) = \begin{cases} \tilde{f}^n_e(\tilde{h}^n_{t(e)}(x^n_{t(e)}),(w^n_\epsilon)_{\epsilon\in I(t(e))}) & \text{if } t(e)\in\mathcal{V}_s \\ \tilde{f}^n_e(x^n_{t(e)}) & \text{otherwise} \end{cases},$$

$$g^n_{k,v}(x^n_v) = \begin{cases} \tilde{g}^n_{k,\tilde{v}_{k,v}}(\tilde{f}^n_{\tilde{e}_{k,v}}(x^n_v)) & \text{if } d(k,v)=1 \\ 0^n & \text{otherwise} \end{cases}.$$

Clearly $C^n=(f^n,g^n)$ emulates $\tilde{C}^n$ and is therefore a solution to $\mathcal{N}$. □

We have shown that if condition (3) is not satisfied, then Construction 3.2 may not preserve achievability of a network. However, as will be clear with our new construction, this condition is not necessary for Theorem 3.1 to hold.

5 New STSD Construction

Construction 5.1 (New STSD Construction). Given an acyclic network $\mathcal{N} = (\mathcal{G}, \mathcal{R}, \mathcal{S}, \mathcal{D})$, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, construct STSD network $\tilde{\mathcal{N}} = (\tilde{\mathcal{G}}, \tilde{\mathcal{R}}, \tilde{\mathcal{S}}, \tilde{\mathcal{D}})$ as follows:

1. Add the following nodes: $v_0, v_1, v_2, \ldots, v_K, \nu_1, \nu_2, \ldots, \nu_K$, and $v_{k,v}, \forall v \in \mathcal{V}_d, k \in \mathcal{K}_v^d$.

2. Add the following edges: $e_k = (v_0, v_k), 1 \leq k \leq K$, $\epsilon_k = (v_k, \nu_k), 1 \leq k \leq K$, $e_{k,v} = (v_k, v), \forall v \in \mathcal{V}_s, k \in \mathcal{K}_v^s$, and $e_{k,v} = (v, v_{k,v})\ \forall v \in \mathcal{V}_d, k \in \mathcal{K}_v^d$.

3. Define $\mathcal{V}_1 = \{v_0\}$, $\mathcal{V}_2 = \{v_k\}_{k=1}^K$, $\mathcal{V}_3 = \{\nu_k\}_{k=1}^K$, $\mathcal{V}_4 = \{v_{k,v} : v \in \mathcal{V}_d, k \in \mathcal{K}_v^d\}$, $\mathcal{E}_1 = \{e_k\}_{k=1}^K$, $\mathcal{E}_2 = \{\epsilon_k\}_{k=1}^K$, $\mathcal{E}_3 = \{e_{k,v} : v \in \mathcal{V}_s, k \in \mathcal{K}_v^s\}$, and $\mathcal{E}_4 = \{e_{k,v} : v \in \mathcal{V}_d, k \in \mathcal{K}_v^d\}$. Set $\tilde{\mathcal{V}} = \mathcal{V} \cup \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \cup \mathcal{V}_4$, $\tilde{\mathcal{E}} = \mathcal{E} \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4$, $\tilde{\mathcal{R}} = [\tilde{r}(e)]_{e \in \tilde{\mathcal{E}}}$, $\tilde{\mathcal{S}} = [\tilde{s}(k,v)]_{1 \leq k \leq K, v \in \tilde{\mathcal{V}}}$, and $\tilde{\mathcal{D}} = [\tilde{d}(k,v)]_{1 \leq k \leq K, v \in \tilde{\mathcal{V}}}$, where

$$\tilde{r}(e) = \begin{cases} r(e) & \text{if } e \in \mathcal{E} \\ 1 & \text{otherwise} \end{cases},$$

$$\tilde{s}(k,v) = \begin{cases} 1 & \text{if } v \in \mathcal{V}_1 \\ 0 & \text{otherwise} \end{cases},$$

$$\tilde{d}(k,v) = \begin{cases} 1 & \text{if } v \in \mathcal{V}_3 \cup \mathcal{V}_4 \\ 0 & \text{otherwise} \end{cases}.$$

The idea of Construction 5.1 is to have v_0 as the single transmitter and nodes in $\mathcal{V}_3$ and $\mathcal{V}_4$ carry single demands. The main difference between Construction 5.1 and Construction 3.2 is that rather than connecting v_0 to the transmitter nodes of $\mathcal{N}$ directly we add intermediate nodes $\{v_k\}_{k=1}^K$ which act as "virtual sources" for each B_k.

Theorem 5.2. *Given any acyclic network $\mathcal{N}$, let $\tilde{\mathcal{N}}$ be the STSD network constructed from $\mathcal{N}$ using Construction 5.1. Then $\mathcal{N}$ is achievable if and only if $\tilde{\mathcal{N}}$ is achievable.*

Proof. If $\mathcal{N}$ is achievable, then there exists a length-n block code $C^n = (f^n, g^n)$ such that

$$g_{k,v}^n(x_v^n) = d(k,v) b_k^n, \quad \forall v \in \mathcal{V}, \forall k, 1 \leq k \leq K.$$

We find a solution to $\tilde{\mathcal{N}}$ by "embedding" (f^n, g^n) into the larger network. Define

$$\tilde{f}_e^n(x_{t(e)}^n) = \begin{cases} b_k^n & \text{if } e = e_k \in \mathcal{E}_1 \\ x_{t(e)}^n & \text{if } e \in \mathcal{E}_2 \cup \mathcal{E}_3 \\ g_{k,t(e)}^n(x_{t(e)}^n) & \text{if } e = e_{k,v} \in \mathcal{E}_4 \\ f_e^n(x_{t(e)}^n) & \text{otherwise} \end{cases},$$

$$\tilde{g}_{k,v}^n(x_v^n) = \begin{cases} x_v^n & \text{if } v = \nu_k \in \mathcal{V}_3 \text{ or } v = v_{k,u} \in \mathcal{V}_4 \\ 0^n & \text{otherwise} \end{cases},$$

In words, the single transmitter sends the k-th information source to v_k, which forwards this information to ν_k as well as to every node that has B_k available in the original network, so that the same edge encoding can be done in the modified STSD network for all $e \in \mathcal{E}$. Any former receiver node then uses the decoding functions of the original solution to send the decoded k-th information source to the corresponding child node ($v_{k,v}$), which simply picks up its input. Clearly, since (f^n, g^n) is a solution to $\mathcal{N}$, $(\tilde{f}^n, \tilde{g}^n)$ must be a solution to $\tilde{\mathcal{N}}$, i.e.

$$\tilde{g}_{k,v}^n(x_v^n) = \tilde{d}(k,v) b_k^n, \quad \forall v \in \tilde{\mathcal{V}}, \forall k, 1 \leq k \leq K.$$

Conversely, if $\tilde{\mathcal{N}}$ has a solution $\tilde{C}^n = (\tilde{f}^n, \tilde{g}^n)$, first we argue that $W^n_{e_k} = \tilde{f}^n_{e_k}(X^n_{v_0}) = \tilde{f}^n_{e_k}(B^n_1, B^n_2, \ldots, B^n_K)$ must be a function of B^n_k only. Noting that $B^n_k = \tilde{g}^n_{\nu_k}(\tilde{f}^n_{\epsilon_k}(X^n_{v_k})) = \tilde{g}^n_{\nu_k}(\tilde{f}^n_{\epsilon_k}(W^n_{e_k}))$, we have

$$n \geq H(W^n_{e_k}) \geq H(\tilde{g}^n_{\nu_k}(\tilde{f}^n_{\epsilon_k}(W^n_{e_k}))) = H(B^n_k) = n,$$

This implies that $H(W^n_{e_k}) = n$, and consequently

$$H(W^n_{e_k}|B^n_k) = H(W^n_{e_k}, B^n_k) - H(B^n_k) = H(W^n_{e_k}) - H(B^n_k) = n - n = 0,$$

i.e. $W^n_{e_k}$ is a function of B^n_k. In other words, the "extra" inputs that a transmitter node v of $\mathcal{N}$ receives in a solution to $\tilde{\mathcal{N}}$ through edges in $\mathcal{E}_3$ must be a function of $(B^n_k)_{k\in\mathcal{K}^s_v}$. Therefore, it is possible to emulate $\tilde{C}^n$ in $\mathcal{N}$ by treating the extra codings of $\tilde{C}^n$ as mappings internal to $\mathcal{N}$'s transmitter and receiver nodes. With slight abuse of notation we now write $\tilde{f}^n_{e_k}(B^n_k)$ in place of $\tilde{f}^n_{e_k}(B^n_1, B^n_2, \ldots, B^n_K)$ and define

$$f^n_e(x^n_{t(e)}) = \begin{cases} \tilde{f}^n_e((\tilde{f}^n_{e_{k,t(e)}}(\tilde{f}^n_{e_k}(b^n_k)))_{k\in\mathcal{K}^s_{t(e)}}, (w^n_\epsilon)_{\epsilon\in I(t(e))}) & \text{if } t(e) \in \mathcal{V}_s \\ \tilde{f}^n_e(x^n_{t(e)}) & \text{otherwise} \end{cases},$$

$$g^n_{k,v}(x^n_v) = \begin{cases} \tilde{g}^n_{k,v}(\tilde{f}^n_{e_{k,v}}(x^n_v)) & \text{if } d(k,v) = 1 \\ 0^n & \text{otherwise} \end{cases},$$

Clearly, (f^n, g^n) emulates $(\tilde{f}^n, \tilde{g}^n)$ in the original network. Since $(\tilde{f}^n, \tilde{g}^n)$ is a solution to $\tilde{\mathcal{N}}$, (f^n, g^n) is a solution to $\mathcal{N}$. □

Note that not only does the above proof not require condition (3) to hold, but it actually makes no assumption about correlation among the B_k's at all. Therefore, with Construction 5.1 and Theorem 5.2 we have shown that Theorem 3.1 holds in general for any acyclic network, where information sources can be arbitrarily correlated.

We apply Construction 5.1 to the example given in Section 4.1 to give more intuition. The resulting network is shown in Figure 3. The idea of the new construction is clearly illustrated: demands from receiver nodes ν_1 and ν_2 force v_1 and v_2 to act as virtual information sources for B_1 and B_2, respectively, making a clear equivalence in achievability between the modified problem and the original one.

Before we close this section, we note, as an example of application, that with Construction 5.1 and Theorem 5.2 the following result is obtained straightforwardly as a generalization of Theorem III.1 of [5], where the same conclusion is drawn for multicast networks.

Theorem 5.3. *Every acyclic network with at most two information sources that has a length-*1 *solution has a length-*1 *linear*[3] *solution.*

We finally note that a somewhat similar construction was used in [6] in proving results for certain constrained classes of network coding problems.

6 Conclusion

We have considered the problem of reducing acyclic network coding problems to equivalent Single-Transmitter-Single-Demand form, first discussed in [1]. We have shown that the prior

[3] In the sense that f^1_e's and g^1_e's are all linear functions over $\mathbb{F}_2$.

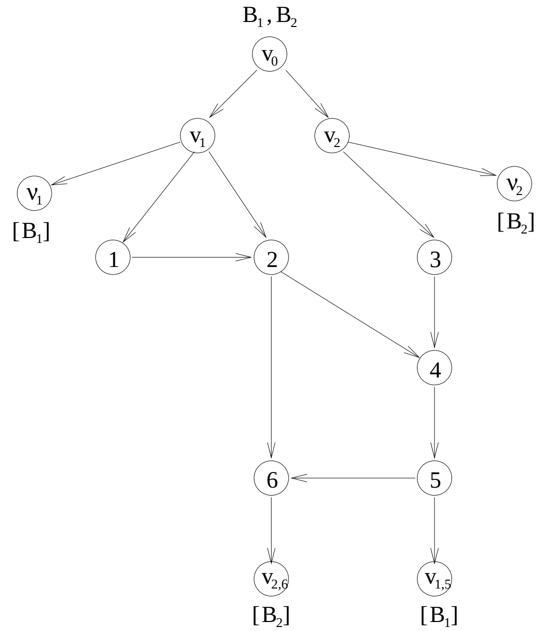

Figure 3: Example: STSD problem constructed using Construction 5.1. Not achievable.

construction does not always yield an equivalent network and requires that information available to different transmitter nodes be independent. We have proposed a new construction and proved that it preserves achievability for any acyclic network. The new construction does not require additional conditions to hold, but works in general for arbitrarily correlated sources. Examples have been shown to support and illustrate the ideas.

The STSD equivalence result enables us to focus on STSD networks without loss of generality. The constraints imposed by the STSD form often make discussions easier. Our results make it possible to apply STSD equivalence to a larger set of network coding problems.

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